A Second Look at Post Crisis Pricing of Derivatives - Part I: A Note on Money Accounts and Collateral

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Abstract

The paper reviews origins of the approach to pricing derivatives post-crisis by following three papers that have received wide acceptance from practitioners as the theoretical foundations for it - [Piterbarg 2010], [Burgard and Kjaer 2010] and [Burgard and Kjaer 2013].

The review reveals several conceptual and technical inconsistencies with the approaches taken in these papers. In particular, a key component of the approach - prescription of cost components to a risk-free money account, generates derivative prices that are not cleared by the markets that trade the derivative and its underlying securities. It also introduces several risk-free positions (accounts) that accrue at persistently non-zero spreads with respect to each other and the risk-free rate. In the case of derivatives with counterparty default risk [Burgard and Kjaer 2013] introduces an approach referred to as semi-replication, which through the choice of cost components in the money account results in derivative prices that carry arbitrage opportunities in the form of holding portfolio of counterparty's bonds versus a derivative position with it.

This paper derives no-arbitrage expressions for default-risky derivative contracts with and without collateral, avoiding these inconsistencies.

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1 Introduction

Derivative transactions today come with an attached collateral account and recognition for the inherent counterparty default risk. The post-crisis paradigm for derivatives pricing dictates that collateralized with cash trades should be discounted using the rate payable on the cash collateral, while a bank's own cost of funding should be used for discounting non-collateralized trades.

Despite the seemingly bifurcated approach to discounting, the two follow a single principle - in both cases the discounting rate is a funding cost rate for the dealer bank (just different funding needs). Moreover, discounting derivatives with the funding cost has been the case for a long time and before the crisis (so the post-crisis paradigm is not really new). For many years (indeed decades) Libor was considered to be the cost at which dealer banks would fund themselves unsecured. Over the years, using this unsecured funding rate as the rate that was plugged in for the risk-free rate in the no-arbitrage pricing formulas for derivatives has given Libor the title of a risk-free rate. This, of course, was more of a misnomer and belief than an economic reality[1]. Nevertheless, as a result of discounting with this single funding cost, dealers would land on the same price, giving the impression that derivatives were priced according to a no-arbitrage pricing approach which upheld the law of one price, and that trades are discounted with a risk-free rate. By implication then one would be forced to state that the inter-dealer trades were happening in a complete and transparent

[1]This misnomer is in fact so prevalent, that the industry and regulatory initiatives for replacing Libor is referred to sometimes as a search for an "alternative risk-free rate".

no-arbitrage market.

Libor, however, was still a cost of unsecured funding for the dealer banks, although it was more of a consensus rate set by the Libor panel of dealers, than a benchmark of market traded securities/rates. This consensus single rate implied that all dealer banks have the same credit quality and cost of funding, were they to fund themselves in capital markets.

The crisis, of course, forced fundamental principles of finance back into practice, whereby the funding cost of each dealer is specific to its balance sheet structure (mix of assets and the capital structure) and quality (earning power of assets). This and the lack of a market mechanism in the inter-dealer market that would guarantee a counterparty on either end of a trade meant that transactions in this market are now akin to lending or borrowing and carry default risk and associated costs.

Although arguments around the inclusion of these costs into derivatives pricing are still ongoing (see for example [Cameron 2013, 2014] and [Hull and White 2012,2016]), and a fundamental approach (or an approach from fundamentals) is still missing, the pricing approach in [Piterbarg 2010], [Burgard and Kjaer 2010] and [Burgard and Kjaer 2013] have become an accepted point of view by major derivatives dealers.

We examine origins of this approach by following the three papers. In Section 2 we examine the treatment of the money account in a general setting. We conclude that no-arbitrage pricing neither provides foundations for, nor supports the structures assigned to the money account, and that in general each structural component of the money account should be a risk-free account accruing at a risk-free rate not to contradict the conditions for no-arbitrage. Section 3 demonstrates that the observations on money accounts apply to the case of default-risky derivatives and, by doing so, derives pricing partial differential equations for default-risky derivatives similar to that of a default-risky bond starting with the setup in [Burgard and Kjaer 2013]. As expected, the bilateral nature of default for derivatives and the derivatives-specific recovery rates emerge as the only difference between pricing a default-risky bond and a default-risky derivative. Section 4 elaborates on these two features and argues that for derivatives to go pari passu with unsecured senior bonds, derivatives have to be priced as fully or partially unsecured liabilities, as opposed to fully collateralized trades. Section 4 also argues that the collateral account cannot be part of the dynamic variables in the replication portfolio or part of the money account with a non-risk-free rate of accrual. Instead, Section 4 introduces collateral as part of an exogenous recovery process, treating it through the collateralization level as a parameter in the valuation formulas for a default-risky derivative. Section 5 concludes with a summary and remarks on the introduction of XVAs within this approach in the future.

2 On the Nature of the Money Account

[Piterbarg 2010] and then [Burgard and Kjaer 2010], [Burgard and Kjaer 2013] introduced an approach for incorporating costs related to collateral and hedging of counterparty default risk into a no-arbitrage (or risk-neutral) price of a derivative. The approach advocated basically amounts to prescribing a structure to the money account, by "taking inspiration" from the costs incurred in managing a typical OTC derivative transaction on a bank's balance sheet. No-arbitrage pricing being the only mechanism (so far) for pricing derivatives neither provides foundations for, nor supports this inspiration. Also implied is that what was known as a position in a risk-free asset (usually proxied with a money account or a risk-free bond) can now be looked at as an account on a bank's balance sheet with components designed to cover the cost structure of a bank's derivatives position, with each component returning bank-specific premiums.

To examine this approach, we start from a most general setup by writing the prescription of a structure to the money account as

$$\rho M = \sum_{a} M_a r_a. \tag{2.1}$$

Here, $M = \{M_a\}_{a=1}^{n}$ are structural components of the money account accruing at different rates r_a, and ρ is the composite rate of return on the overall money account M that follows by construction. Formally, the results in [Piterbarg 2010], [Burgard and Kjaer 2010], [Burgard and Kjaer 2013] can be obtained by specifying terms on the right hand side of Eqn.2.1 following cost structures in these papers (See Appendix A for details.).

So what can be said about the nature of structural components prescribed to the money account, the accrual rates on these money account components, and the overall rate of return on the money account itself.

We start with the standard Black-Scholes-Merton no-arbitrage pricing setup ("BSM") ([Merton 1973] and [Black and Scholes 1973]) as our working model. A self-financing arbitrage portfolio Π is set up with a position h_V in the derivative V and its replicating portfolio. The latter consists of a position h_S in a δ dividend paying underlying security S and a position in a risk-free security, proxied with a money account M. Using the general form of the money account distribution in Eqn.2.1

instantaneous return on such a portfolio when it is delta-hedged is given by

$$\begin{aligned} d\Pi^{\tilde{h}} &= \left[\mu_V V - (\mu+\delta)\frac{\sigma_V}{\sigma}V\right] h_V dt + dM \\ &= \left[\mu_V V - (\mu+\delta)\frac{\sigma_V}{\sigma}V\right] h_V dt + \sum_a dM_a \\ &= \left[\mu_V V - (\mu+\delta)\frac{\sigma_V}{\sigma}V\right] h_V dt + \sum_a M_a r_a dt \\ &= \left[\mu_V V - (\mu+\delta)\frac{\sigma_V}{\sigma}V\right] h_V dt + \rho M dt. \end{aligned} \tag{2.2}$$

Here, being delta-hedged has its usual meaning - i.e., special weights $\tilde{h}_S = -\frac{\sigma_V}{\sigma}\frac{V}{S}h_V$ and $\tilde{h}_V = h_V$ exist in the market $\{V, S, M\}$. Notice, that $\tilde{h}_S \neq -\frac{\partial V}{\partial S}h_V \equiv \Delta h_V$, since we have not yet assumed that V = V(t, S(t)). We have only assumed that both the derivative and the underlying security price the same risk factor in all states of the world.

With a zero initial investment (***z.i.i. constraint*** henceforth), one has

$$\Pi^{\tilde{h}}(t) = 0 \Rightarrow M = -\tilde{h}_S S - \tilde{h}_V V = \left(\frac{\sigma_V}{\sigma} - 1\right) h_V V := \sum_a M_a, \tag{2.3}$$

leading to the following form of Eqn.2.2

$$d\Pi^{\tilde{h}} = \left[\mu_V V - (\mu+\delta)\frac{\sigma_V}{\sigma}V + \left(\frac{\sigma_V}{\sigma} - 1\right) V\rho\right] h_V dt. \tag{2.4}$$

Requiring the no-arbitrage prices to exist

$$\mathbb{P}\left[\Pi^{\tilde{h}}(t+dt)\right] = \mathbb{P}\left[\Pi^{\tilde{h}}(t) \mid_{\Pi^{\tilde{h}}(t)=0} + d\Pi^{\tilde{h}}\right] = \mathbb{P}\left[d\Pi^{\tilde{h}} = 0\right] = 1, \tag{2.5}$$

is equivalent to setting the square brackets in Eqn. 2.4 to zero. This yields the following risk-return condition for the traded in this market derivatives and their underlying securities

$$\frac{\mu_V - \rho}{\sigma_V} = \frac{(\mu+\delta) - \rho}{\sigma}. \tag{2.6}$$

Eqn.2.6 is the condition for no-arbitrage prices to exist in the $\{V, S, M\}$ market. It is a more fundamental relationship than its BSM PDE representation. To arrive to a BSM PDE representation of Eqn.2.6 and its Feynman-Kač solution, one still has to make two essential assumptions - 1) $V = V(t, S(t))$ and 2) $V(T, S(T)) = Contractual\ Payoff \equiv \Phi(T, S(T))$. After this, using Ito's Lemma produces a PDE form of the Eqn.2.6

$$\frac{\partial V}{\partial t} + (\rho - \delta) S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \rho V = 0. \tag{2.7}$$

The only difference here with the classical BSM equation is the composite rate of return $\rho = \sum_a \frac{M_a}{M} r_a$ for the money account.

In this sense, the no-arbitrage setup itself does not fix the rate of accrual for the money account. It is rather the risk-free nature of the money account that dictates that its rate of accrual should be the risk-free rate (i.e., $\rho = r$) to avoid arbitrage between risk-free securities (accounts), that accrue at premium differentials to each other.

In XVA literature money accounts are sometimes constructed bottom-up. First, its components are defined and accrual rates are assigned according to some costs structure arguments (or inspirations). Then, the sum of these component money accounts is defined as the money account - $\sum_a M_a := M$. It should be reminded, however, that the cost structure is usually outside of the replication framework in the market, and for the z.i.i. constraint for a delta-hedged portfolio $\Pi^{\tilde{h}}$ to hold

$$\Pi^{\tilde{h}} = h_V V - \Delta h_V S + M = 0, \; i.e., \; M = -h_V V + \Delta h_V S = (\Delta S - V)\, h_V, \tag{2.8}$$

the $\{V, S, M\}$ market has to clear

$$-\frac{h_V V}{M} + \frac{\Delta h_V S}{M} = 1 = \sum_a w_a, \; where \; w_a = \frac{M_a}{M}. \tag{2.9}$$

In [Piterbarg 2010], for example, this is maintained by adding and subtracting the collateral account for collateralized derivatives (with $h_V = -1$)

$$\frac{(V - C + C)}{M} - \frac{\Delta S}{M} = \frac{(V - C)}{M} - \frac{C}{M} - \frac{\Delta S}{M} = \sum_a w_a = 1. \tag{2.10}$$

Although this can be looked at as a mathematical identity, it implies replication of the difference between a derivative V and its collateral account C with a position in the underlying and a risk-free asset

$$V - C = \Delta S + (M - C) = \Delta S + M', \tag{2.11}$$

which does not follow from any fundamental statement about replicability of a contingent claim in this market (collateral prices no risk factor).

More importantly, the z.i.i. constraint Eqn.2.8, which implied by the feasible replicability relationships in a given market, defines the size of the money account M and it does not prescribe a structure M_a to it (i.e., z.i.i. constraint holds for the total of the money account).

To study Eqn.2.6 further, let's rewrite the instantaneous return Eqn.2.4 for the delta-hedged self-financing arbitrage portfolio $\Pi^{\tilde{h}}$ by combining the general form for the money account structure Eqn.2.1 and the z.i.i. constraint Eqn.2.3

$$d\Pi^{\tilde{h}} = \left[\frac{\mu_V - r}{\sigma_V} - \frac{\mu + \delta - r}{\sigma} + \frac{\left(\frac{\sigma_V}{\sigma} - 1\right) \sum_a w_a (r_a - r)}{\sigma_V} \right] h_V V \sigma_V dt. \tag{2.12}$$

Requiring for the arbitrage-free prices to exist (i.e., Eqn.2.5 to hold), leads to the following form of the fundamental relation Eqn.2.6

$$\frac{\mu_V - r}{\sigma_V} + \frac{\left(\frac{\sigma_V}{\sigma} - 1\right) \sum_a w_a (r_a - r)}{\sigma_V} = \frac{\mu + \delta - r}{\sigma}. \tag{2.13}$$

The extra premium for the same level of risk σ_V on the left hand side of Eqn.2.13 is due to non-zero premiums $r_a - r = \epsilon_a \geq 0$ embedded in the (risk-free) components of the money account. It states that extra premium(s) can be earned by holding the derivative instead of a position in the underlying.[2]. In other words, one hedges all of the σ_V in the market for the underlying, get compensated per unit of derivative position hedged through

$$\mu_V = r + (\mu + \delta - r) \frac{\sigma_V}{\sigma} = r + (\mu + \delta - r) \left| \frac{\tilde{h}_S S}{h_V V} \right|_{\tilde{h}_S = -\Delta h_V} = r + (\mu + \delta - r) \left| \frac{\Delta S}{V} \right|,$$

and still makes extra premiums on top of a risk-free rate of return, despite the fact that at that point it is a risk free portfolio itself (i.e., with weights $\tilde{h}_S = -\Delta h_V$).

One can also look at the Eqn.2.12 slightly differently (leading to the same argument). No-arbitrage pricing is effectively a proof that no-arbitrage prices exist in a given market, as opposed to a statement that all securities in that market are traded at no-arbitrage prices. However, if an equilibrium is attainable in the complete market $\{V, S, M\}$ (i.e., all agents have been able to attain thier optimal risk-return balances), then one should put

$$\frac{\mu_V - r}{\sigma_V} = \frac{\mu + \delta - r}{\sigma}, \tag{2.14}$$

and one could state that markets have cleared at no-arbitrage prices.

[2]One could argue that a derivative position on a security is always more volatile (more risky) than an outright position in the underlying security and so the price of a derivative should always be greater or equal to the price of the underlying. This would make the derivative a dominant security ([Merton 1973]), however, the dominance cannot be due to dealer-specific cost structure add-ons to the market clearing derivative prices.

The two arguments above imply that for the no-arbitrage prices to clear in the complete market $\{V, S, M\}$, the second term on the left hand side of Eqn.2.13 should be zero with certainty

$$\mathbb{P}[d\Pi^{\tilde{h}} = \left[\frac{\left(\frac{\sigma_V}{\sigma} - 1\right)\sum_a w_a(r_a - r)}{\sigma_V}\right] h_V V dt = 0] = 1, \tag{2.15}$$

leading to

$$r_a - r = 0,\ for\ all\ a\ , and\ \rho = r. \tag{2.16}$$

Notice that we have not made an a priori assumption that $\rho = r$ in the derivations of Eqns.2.13 - 2.15.

In other words, if one has assumed that M_a structural components of the money account are risk-free, then all components M_a must accrue at the risk-free rate r to avoid arbitrage - several portfolios presumed risk-free are returning at persistently non-zero premiums with respect to each other.

Although we refer to Eqn.2.12 and Eqn.2.13 as arbitrage, the more accurate statement would be that the market $\{V, S, M\}$ does not clear the equilibrium (no-arbitrage) relation Eqn.2.13

$$\frac{\mu_V - r}{\sigma_V} + \frac{\left(\frac{\sigma_V}{\sigma} - 1\right)\sum_a w_a(r_a - r)}{\sigma_V} \neq \frac{\mu + \delta - r}{\sigma}. \tag{2.17}$$

Eqn.2.13 would point at arbitrage if the term $\frac{\left(\frac{\sigma_V}{\sigma} - 1\right)\sum_a w_a(r_a - r)}{\sigma_V}$ on its left hand side was generated by a market-priced security in the $\{V, S, M\}$ market (as part of the replicating portfolio).

The correct equilibrium (no-arbitrage) expression is Eqn.2.14, which states that there is no extra unit of risk-adjusted premium to be received for holding a derivative position, compared to the outright holding of a position in the underlying, if the markets are arbitrage-free (and complete).

This does not prohibit from choosing a benchmark $\rho = r + \epsilon$ to compare the returns from holding a position in the derivative - $\frac{\mu_V - \rho}{\sigma_V}$, or a position in the underlying - $\frac{\mu + \delta - \rho}{\sigma}$, and still write

$$\frac{\mu_V - r - \epsilon}{\sigma_V} = \frac{\mu + \delta - r - \epsilon}{\sigma}. \tag{2.18}$$

This simply means the returns over a chosen benchmark ρ are lesser exactly by the amount of $\epsilon = \rho - r$ (investor specific cost above the risk-free rate). The latter has no reflection on the risk-return equilibria in the market $\{V, S, M\}$.

To summarize observations above, the money account can have any number of structural components as long as

(i) each component of the money account is a risk-free account,

(ii) each component accrues at a risk-free rate, and

(iii) all these risk-free rates are the same.

Observations above and the structure of the arbitrage portfolio Π also imply that a collateral account cannot be part of the replicating portfolio either as part of the risky positions (the hedge) because it prices no risk factor, or the money account because it's accruing at a non-risk-free rate.

In the next section we show that observations (i) - (iii) also hold for the case of default-risky derivatives, and derive pricing equations for the default-risky derivatives with and without collateral. We show that the money account in this case is simply a bigger account compared to the case of non-default-risky derivatives pricing of BSM.

We mention in passing, that in the case of no risk-free assets in the market the money account and its components can only be zero-beta portfolios, accruing at the same single rate of return for a zero-beta portfolio ([Black 1972]). Observations (i) - (iii) also hold true with respect to zero-beta portfolios.

3 Pricing Derivatives with Counterparty Risk

We start from the widely accepted assumption in the practitioner literature that the value of a traded derivative depends on the bi-lateral default risk of the counterparties involved

$$\hat{V} = \hat{V}(t; S(t), J_A(t), J_B(t)). \tag{3.1}$$

The setup[3] of a self-financing arbitrage portfolio Π^h in this case consists of $h_{\hat{V}}$ quantity of a derivative instrument $\hat{V}$ on a security S, with its replicating portfolio consisting of h_S quantity of the security, h_B quantity of risky bonds P_B issued by the counterparty B, h_A quantity of risky bonds P_A issued by counterparty A and an amount M of money account[4]. The value of this portfolio at time t is

$$\Pi^h(t) = h_S S(t) + h_{\hat{V}} \hat{V}(t) + h_A P_A(t) + h_B P_B(t) + M(t), \tag{3.2}$$

with the instantaneous return given in terms of gain processes as

$$d\Pi^h = h_S dG_S + h_{\hat{V}} dG_{\hat{V}} + h_A dG_A + h_B dG_B + dG_M. \tag{3.3}$$

The gain dynamics of the portfolio positions are defined as

$$\begin{aligned} dG_S &= dS + dD_S = dS + \delta S dt = (\mu + \delta) S dt + \sigma S dz \\ dG_{\hat{V}} &= d\hat{V} + 0 = \mu_{\hat{V}} \hat{V} dt + \sigma_{\hat{V}} \hat{V} dz + \Delta \hat{V}_A dJ_A + \Delta \hat{V}_B dJ_B \\ dG_A &= dP_A + r_A P_A dt = -(1 - R_A) P_A^- dJ_A + r_A P_A dt, \\ dG_B &= dP_B + r_B P_B dt = -(1 - R_B)_B dJ_B + r_B P_B dt, \\ dG_M &= 0 + dD_M = \rho M dt. \end{aligned} \tag{3.4}$$

Here, as in the previous section, we have taken M to be a money account with generic components M_a and a composite accrual rate ρ as in Eqn.2.1. R_A and R_B are the recovery rates for the senior unsecured bonds issued by counterparties A and B respectively. It is assumed that the bonds are senior unsecured debentures to discuss the fact that derivatives go pari passu with this type of debt in default. In general, bonds P_A and P_B can be any debentures issued by the counterparty, since what is being replicated is the default risk, while the recovery levels drive the hedge ratios for full coverage of losses in case of default.

$\Delta \hat{V}_B$ and $\Delta \hat{V}_A$ are changes in the value of the derivative position due to default by the counterparty B and counterparty A, respectively:

$$\begin{aligned} \Delta \hat{V}_B &= \hat{v}_B - \hat{V}(t; S, dJ_A = 0, dJ_B = 0), \\ \Delta \hat{V}_A &= \hat{v}_A - \hat{V}(t; S, dJ_A = 0, dJ_B = 0), \end{aligned} \tag{3.5}$$

[3] J_A and J_B are indicators of default for counterparty A and B respectively, and take value 1, if the corresponding counterparty is in default and zero otherwise.

[4] Here we are following the setup in the XVA literature that originates from [Burgard and Kjaer 2010]. Fundamentally, a bilateral default is replicated through a first-to-default CDS on the counterparties, since recovery is triggered by the first default event by either of the parties.

with $\hat{v}_A$ and $\hat{v}_B$ the residual values of the risky derivative $\hat{V}$, when the counterparty A or counterparty B is in default, respectively

$$\begin{aligned} \hat{v}_B(u) &:= \hat{V}(t; S, dJ_A = 0, dJ_B = 1), \\ \hat{v}_A(u) &:= \hat{V}(t; S, dJ_A = 1, dJ_B = 0). \end{aligned} \tag{3.6}$$

Plugging Eqn.3.4 into the instantaneous return equation Eqn.3.3 yields to the following

$$\begin{aligned} d\Pi^h &= \left[h_S(\mu+\delta)S + h_{\hat{V}}\mu_{\hat{V}}V\right]dt + \left[h_B r_B P_B + h_A r_A P_A\right]dt + \rho M dt \\ &+ \left[h_S \sigma S + h_{\hat{V}}\sigma_{\hat{V}}V\right]dz \\ &+ (1-R_B)\left[h_{\hat{V}}\hat{V}\frac{\Delta\hat{V}_B}{\hat{V}(1-R_B)} - h_B P_B\right]dJ_B \\ &+ (1-R_A)\left[h_{\hat{V}}\hat{V}\frac{\Delta\hat{V}_A}{\hat{V}(1-R_A)} - h_A P_A\right]dJ_A. \end{aligned} \tag{3.7}$$

Observe now, that $\frac{\Delta\hat{V}_A}{\hat{V}}$ and $\frac{\Delta\hat{V}_B}{\hat{V}}$ are negative numbers and are percentage drops in the value of the derivative position due to defaults by A and B, respectively. Hence, **define recovery rates χ_B and χ_A for the derivative position** as

$$\frac{\Delta\hat{V}_B}{\hat{V}} = -(1-\chi_B),\ and\ \frac{\Delta\hat{V}_A}{\hat{V}} = -(1-\chi_A).$$

We further introduce **loss ratios z_A and z_B - the ratio of loss rate from a derivative position to loss rate from a bond position of a counterparty**

$$z_A = \frac{1-\chi_A}{1-R_A},\ and\ z_B = \frac{1-\chi_B}{1-R_B}.$$

In other words, we are modeling recovery rates from the derivative position as a percentage of the value of the default-risky derivative prior to default

$$\hat{v}_A = \chi_A\hat{V}(t; S, dJ_A = 0, dJ_B = 0), \tag{3.8}$$
$$\hat{v}_B = \chi_B\hat{V}(t; S, dJ_A = 0, dJ_B = 0). \tag{3.9}$$

Rewriting Eqn.3.7 with these new notations one arrives to

$$\begin{aligned} d\Pi^h &= \left[h_S(\mu+\delta)S + h_{\hat{V}}\mu_{\hat{V}}V + h_B r_B P_B + h_A r_A P_A\right]dt + M\rho dt \\ &+ \left[h_S \sigma S + h_{\hat{V}}\sigma_{\hat{V}}V\right]dz \\ &+ (1-R_B)\left[-h_{\hat{V}}\hat{V}z_B - h_B P_B\right]dJ_B \\ &+ (1-R_A)\left[-h_{\hat{V}}\hat{V}z_A - h_A P_A\right]dJ_A. \end{aligned} \tag{3.10}$$

If the market $\left\{\hat{V}, S, P_A, P_B, M\right\}$ clears, then the **the z.i.i. constraint**

$$\Pi^h = h_S S + h_{\hat{V}}\hat{V} + h_B P_B + h_A P_A + M = 0, \tag{3.11}$$

is a constraint on portfolio weights h such that satisfy

$$-h_S^{'} S - h_{\hat{V}}^{'}\hat{V} - h_B^{'} P_B - h_A^{'} P_A = M. \tag{3.12}$$

Plugging into Eqn.3.10 one arrives to the following expression for the instantaneous return on such portfolio

$$\begin{aligned} d\Pi^{h^{'}} &= \left[h_S^{'}(\mu+\delta-\rho)S + h_{\hat{V}}^{'}\mu_{\hat{V}}V\right]dt \\ &+ \left[h_B^{'}(r_B-\rho)P_B + h_A^{'}(r_A-\rho)P_A\right]dt \\ &+ \left[h_S^{'}\sigma S + h_{\hat{V}}^{'}\sigma_{\hat{V}}V\right]dz \\ &+ (1-R_B)\left[-z_B h_{\hat{V}}^{'}\hat{V} - h_B^{'}P_B\right]dJ_B \\ &+ (1-R_A)\left[-z_A h_{\hat{V}}^{'}\hat{V} - h_A^{'}P_A\right]dJ_A. \end{aligned} \tag{3.13}$$

It is easy to see now, that a portfolio strategy[5]

$$\tilde{h} = \left\{-\Delta h_{\hat{V}}^{'}, h_{\hat{V}}^{'}, -z_B\frac{\hat{V}}{P_B^-}h_{\hat{V}}^{'}, -z_A\frac{\hat{V}}{P_A^-}h_{\hat{V}}^{'}, 1\right\} \tag{3.14}$$

that satisfies the z.i.i. constraint Eqn.3.11

$$M = \left(\Delta S - \hat{V} + z_B\hat{V} + z_A\hat{V}\right)h_{\hat{V}}^{'}, \tag{3.15}$$

will make instantaneous return on such portfolio equal to

$$d\Pi^{h^{'}}\mid_{h^{'}=\tilde{h}} = \left[(\mu_{\hat{V}}-\rho)\hat{V} - \Delta(\mu+\delta-\rho)S - z_B(r_B-\rho)\hat{V} - z_A(r_A-\rho)\hat{V}\right]h_{\hat{V}}^{'}dt \tag{3.16}$$

with certainty.

We rewrite the delta-hedging weights explicitly for a later use

$$\tilde{h}_S = -\Delta h_{\hat{V}}^{'},\ and\ \tilde{h}_{\hat{V}} = h_{\hat{V}}^{'}; \tag{3.17}$$

[5]We have used the fact that $\hat{V} = \hat{V}(t; S(t), J_A(t), J_B(t))$ and so $\frac{\sigma_{\hat{V}}}{\sigma}\frac{\hat{V}}{S} = \frac{\partial\hat{V}}{\partial S} \equiv \Delta$.

$$\tilde{h}_A = -z_A \frac{\hat{V} h'_{\hat{V}}}{P_A} = -\frac{(1-\chi_A)}{(1-R_A)} \frac{\hat{V} h'_{\hat{V}}}{P_A}, \tag{3.18}$$

$$\tilde{h}_B = -z_B \frac{\hat{V} h'_{\hat{V}}}{P_B} = -\frac{(1-\chi_B)}{(1-R_B)} \frac{\hat{V} h'_{\hat{V}}}{P_B}. \tag{3.19}$$

Satisfying the no-arbitrage conditions

$$\mathbb{P}\left[\Pi^{\tilde{h}}(t+dt)\right] = \mathbb{P}\left[\Pi^{\tilde{h}}(t) \mid_{\Pi^{\tilde{h}}(t)=0} + d\Pi^{\tilde{h}}\right] = \mathbb{P}\left[d\Pi^{\tilde{h}} = 0\right] = 1 \tag{3.20}$$

is now equivalent to setting

$$(\mu_{\hat{V}} - \rho)\hat{V} - \Delta(\mu + \delta - \rho)S - z_B(r_B - \rho)\hat{V} - z_A(r_A - \rho)\hat{V} = 0. \tag{3.21}$$

If one assumes that the market $\left\{\hat{V}, S, P_A, P_B, M\right\}$ is frictionless (no material liquidity premiums), complete and risk-neutrally priced, then

$$\mathbb{E}\left[dJ_A\right] \approx \lambda_A dt \; and \; \mathbb{E}\left[dJ_B\right] \approx \lambda_B dt, \; with \tag{3.22}$$

$$r_A - r = (1-R_A)\lambda_A \; and \; r_B - r = (1-R_B)\lambda_B, \tag{3.23}$$

and following the arguments in Section 2 for Eqns.2.13 - 2.15, $\rho - r = \sum_a w_a(r_a - r) = 0$.

This leads to the following form of the fundamental no-arbitrage relationship Eqn.2.6 for the case of default-risky derivatives[6]

$$\frac{\mu_{\hat{V}} - \left[(1-\chi_A)\lambda_A + (1-\chi_B)\lambda_B\right] - r}{\sigma_V} = \frac{(\mu+\delta) - r}{\sigma}, \; or$$

$$\frac{\mu_V - r}{\sigma_V} = \frac{(\mu+\delta) - r}{\sigma}, \tag{3.24}$$

where $\mu_V = \mu_{\hat{V}} - \left[(1-\chi_A)\lambda_A + (1-\chi_B)\lambda_B\right]$ is the expected return on an otherwise identical default-risk-free derivative $V(t, S(t))$.

Eqn.3.24 is a no-arbitrage condition for the market $\left\{\hat{V}, S, P_A, P_B, M\right\}$ and it states that if default risk between two default-risky counterparties can be replicated through bilateral exchange of bonds that are also traded in the market (mutually shorting bonds), then **after a market-priced adjustment for the bilateral default risk** there is no extra unit of risk-adjusted premium to be

[6]Notice that $\lambda_{A,B}$ are not the same as in [Burgard and Kjaer 2010], as the rate of return on collateral account r_C is not the risk-free rate - $r_C \neq r$.

obtained for holding a default-risky derivative position, instead of holding an outright position in the underlying, if the markets are arbitrage-free (and complete).

In contrast, prescribing a structure to the money account M, using the anzats Eqn.2.1 leads to

$$\begin{aligned}\frac{\mu_{\hat{V}}-\left[\left(1-\chi_A\right)\lambda_A+\left(1-\chi_B\right)\lambda_B\right]-r}{\sigma_V} &+ \frac{\left[\left(\frac{\sigma_V}{\sigma}-1\right)+\frac{1-\chi_A}{1-R_A}+\frac{1-\chi_B}{1-R_B}\right]\sum_a w_a(r_a-r)}{\sigma_V} \\ &\neq \frac{(\mu+\delta)-r}{\sigma},\ or \\ \frac{\mu_V-r}{\sigma_V} &+ \frac{\left[\left(\frac{\sigma_V}{\sigma}-1\right)+\frac{1-\chi_A}{1-R_A}+\frac{1-\chi_B}{1-R_B}\right]\sum_a w_a(r_a-r)}{\sigma_V} \\ &\neq \frac{(\mu+\delta)-r}{\sigma},\end{aligned} \tag{3.25}$$

which does not hold for the same reasons discussed in Section 2 for the case of no default risk.

The market $\left\{\hat{V},S,P_A,P_B,M\right\}$ simply does not clear no-arbitrage prices of a derivative and its replicating portfolio in a way that Eqn.3.25 between the risk premia in that market would hold.

Eqn.3.24 can also be written in a PDE presentation using Ito's Lemma for $\hat{V}\mu_{\hat{V}}$ and $\hat{V}\sigma_{\hat{V}}$ in the standard form with the assumption Eqn.3.1

$$\mathcal{L}^{(r-\delta)}\hat{V}-r\hat{V}=\left[(1-\chi_B)\lambda_B+(1-\chi_A)\lambda_A\right]\hat{V}, \tag{3.26}$$

where, for the brevity of expressions, we have adopted the notation[7]

$$\mathcal{L}^{(x-y)}\hat{V}:=\frac{\partial\hat{V}}{\partial t}+(x-y)\,S\frac{\partial\hat{V}}{\partial S}+\frac{1}{2}\sigma^2S^2\frac{\partial^2\hat{V}}{\partial S^2}. \tag{3.27}$$

With the appropriate terminal conditions one can write a Feynman-Kač solution for Eqn.3.26 as follows

$$\begin{aligned}\hat{V}(t,S(t),J_A(t),J_B(t)) &= \mathbb{E}^Q\left[Z_r(t,T)\Phi(T,S(T))\right] \\ &- \mathbb{E}^Q\left[Z_r(t,u)\left(1-\chi_B\right)\lambda_B(u)\hat{V}(T-(t+u),S(u))du\right] \\ &- \mathbb{E}^Q\left[Z_r(t,u)\left(1-\chi_A\right)\lambda_A(u)\hat{V}(T-(t+u),S(u))du\right],\end{aligned} \tag{3.28}$$

[7] Formally, one could still write a PDE form for the expression Eqn.3.25

$$\mathcal{L}^{(\rho-\delta)}\hat{V}-\rho\hat{V}=\left[z_B(r_B-\rho)+z_A(r_A-\rho)\right]\hat{V},$$

but its solution will not represent a no-arbitrage price of a derivative that is cleared by a complete market.

with the simplified notation $Z_r(t,T) = e^{-\int_t^T r(\tau)d\tau}$.

Eqn.3.26 can also be re-written as

$$\mathcal{L}^{(r-\delta)}\hat{V} - [r + (1-\chi_B)\lambda_B + (1-\chi_A)\lambda_A]\,\hat{V} = 0, \tag{3.29}$$

resulting in a different form of the Feynman-Kač solution

$$\hat{V}(t, S(t), J_A(t), J_B(t)) = \mathbb{E}^Q\left[e^{-\int_t^T [r(\tau)+(1-\chi_B)\lambda_B(\tau)+(1-\chi_A)\lambda_A(\tau)]d\tau}\,\Phi(T, S(T))\right]. \tag{3.30}$$

The following observations are in order.

(A). Eqn.3.26 and Eqn.3.29 that price default risky bonds (see, for example [Duffie and Singleton 1999]) are also the equations that price default risky derivatives. Notice, that only contractual cash-flows enter the expressions. Differences with pricing a risky bond come only from

- the bi-lateral nature of the default event[8], and
- the recovery rate for a derivative.

(B). The money account in the case of default-risky derivatives is simply a bigger account compared to the case of non-default-risky derivatives pricing of BSM

$$\begin{aligned} M = \left(\Delta S - \hat{V} + z_B\hat{V} + z_A\hat{V}\right) h'_{\hat{V}} \quad &= \quad \left(\Delta S - \hat{V}\right) h'_{\hat{V}} + \left(z_B\hat{V} + z_A\hat{V}\right) h'_{\hat{V}} \\ &:= M_{BSM} + M_{Defaultrisk}, \end{aligned} \tag{3.31}$$

where $M_{BSM} = \left(\Delta S - \hat{V}\right) h'_{\hat{V}}$ is the money account from the BSM pricing of non-default-risky derivatives and $M_{Defaultrisk} = \left(z_B\hat{V} + z_A\hat{V}\right) h'_{\hat{V}}$ is an additional money account component due to the default risk, both accruing at the risk-free rate

$$rM = rM_{BSM} + rM_{Defaultrisk}. \tag{3.32}$$

In the language of Section 2 one has a composite accrual rate for the money account $\rho = \frac{M_{BSM}}{M} r_{BSM} + \frac{M_{Defaultrisk}}{M} r_{Defaultrisk}$, but with $r_{BSM} = r_{Defaultrisk} = r$, so it satisfies $\rho - r = \sum_a w_a(r_a - r) = 0$.

In other words, - **they are of the same risk-free nature as the money account as a whole, and are accruing at the same risk-free rate**.

[8] See also [Duffie and Huang 1996] and the references therein for the bi-lateral nature of default risk in derivatives.

In the next section we take a closer look at the two main differences between the pricing PDE Eqn.3.26 and its bond counterparts - the bi-lateral nature of default and recovery rates for derivative positions. We also demonstrate how collateral enters the pricing equations as a parameter, consistent with the observations above that collateral cannot be part of the replicating portfolio either as part of the hedging positions because it prices no risk factor, or the money account because it's accruing at a non-risk-free rate.

4 Bilateral Default, Recovery Rates for Derivatives and Collateral

While the default risk between two trading counterparties is always bilateral in nature, the price of the derivative traded does not have to be. Starting from an assumption that the price of a derivative depends on the bilateral default risk of the counterparties involved (Eqn.3.1), presumes that in general a derivative (its contractual payoff function) may generate liability cashflows for both counterparties. However, if a) the payoff function (contractual cashflows) of a derivative gives rise to liability cashflows to only one of the counterparties, or b) the liability cashflows of counterparties are separable with certainty (i.e., the payoff function is linear in the two streams of liability cashflows), then Eqn.3.26 translates into separate unilateral default-risky equations for the counterparties A and B

$$\mathcal{L}^{(r-\delta)}\hat{V}_{A,B} - r\hat{V}_{A,B} = (1-\chi_{A,B})\lambda_{A,B}\hat{V}_{A,B}. \tag{4.1}$$

Here $\hat{V_A}$ and $\hat{V_B}$ are the prices for derivatives, the payoff function for which are the liability cashflows due from the counterparty A - $\Phi_A(T,S(T))$ and counterparty B - $\Phi_B(T,S(T))$, respectively[9].

In general, where separability of liability cashflows cannot be achieved, one needs a more general form of the Eqn.3.26

$$\mathcal{L}^{(r-\delta)}\hat{V} - r\hat{V} = \lambda_B\left(\hat{V}-\hat{v}_B\right) + \lambda_A\left(\hat{V}-\hat{v}_A\right). \tag{4.2}$$

Here $\hat{v}_A$ and $\hat{v}_B$ are the residual values of the (same) derivative contract when counterparty A or counterparty B is in default respectively, as defined in Eqn.3.6. Hence, we rewrite the Feynman-Kač solution for Eqn.4.2 as

$$\begin{aligned} \hat{V}(t,S(t),J_A(t),J_B(t)) &= \mathbb{E}^Q\left[Z_r(t,T)\Phi(T,S(T))\right] \\ &- \mathbb{E}^Q\left[\int_t^T Z_r(t,u)\lambda_B(u)\left(\hat{V}(u)-\hat{v}_B(u)\right)du\right] \\ &- \mathbb{E}^Q\left[\int_t^T Z_r(t,u)\lambda_A(u)\left(\hat{V}(u)-\hat{v}_A(u)\right)du\right]. \end{aligned} \tag{4.3}$$

Estimation of $\hat{v}_A$ and $\hat{v}_B$ depends on the applicability and existence of one of the two mechanisms - **replacement, or recovery**.

In a replacement paradigm the following two cases are of interest.

<u>Case 1.</u> Assume there is a market mechanism for providing counterparty replacement (both for defaulted and solvent counterparties) and all market participants pay the same price for a given

[9] The trade can be treated as a structured deal, each component priced as a unilateral default-risky derivative - no need for considering bi-lateral expressions.

contractual stream of cashflows (similar to the pre-crisis setup).

In this case, the derivative contract $\hat{V}(u) = \hat{V}(T-(t+u), S(u))$ at any intermediate time $u \in [t,T]$ can be replaced by an identical one with any other counterparty during the life of the derivative

$$\hat{v}_A(u) = \hat{v}_B(u) = \hat{V}(u) = \hat{V}(T-(t+u), S(u)). \tag{4.4}$$

The default-risk terms in the Feynman-Kač solution Eqn.4.3 vanish, leading to

$$\hat{V}(t, S(t), J_A(t), J_B(t)) = \mathbb{E}^Q\left[Z_r(t,T)\Phi(T, S(T))\right] = V(t, S(t)). \tag{4.5}$$

This outcome is equivalent (at least mechanically) to relaxing the assumption that the price of a derivative that promises a given stream of cashflow depends on the default risk of the counterparties trading it.

The experience of the 2007-2009 financial crisis showed that assuming an a priori existence of a market mechanism for replacement can be a material presumption.

Case 2. Assume there is no market mechanism for providing a replacement counterparty, but it is possible to replace the derivative contract with an identical one with another solvent counterparty C, and the price at which this replacement is available depends (bilaterally) on the default risk of the parties involved (similar to the post-crisis setup).

If $\hat{V}_{X,Y}$ is the value of a default-risky derivative contract that pays $\Phi(T, S(T))$ at maturity between two solvent counterparties X and Y, then one can set

$$\begin{aligned} \hat{V}(u) &= \hat{V}(u; S, dJ_A = 0, dJ_B = 0) := \hat{V}_{A,B}(u), \ \textit{as a notation}; \qquad (4.6)\\ \hat{v}_B(u) &= \hat{V}(u; S, dJ_A = 0, dJ_B = 1) := \hat{V}_{A,C}(u), \ \textit{as a replacement when } B \textit{ is in default};\\ \hat{v}_A(u) &= \hat{V}(u; S, dJ_A = 1, dJ_B = 0) := \hat{V}_{B,C}(u), \ \textit{as a replacement when } A \textit{ is in default}. \end{aligned}$$

With these notations one then has for the Feynman-Kač solution Eqn.4.3

$$\begin{aligned} \hat{V}(t; S(t), J_A(t), J_B(t)) &= \mathbb{E}^Q\left[Z_r(t,T)\Phi(T, S(T))\right] \qquad (4.7)\\ &- \mathbb{E}^Q\left[\int_t^T Z_r(t,u)\lambda_B(u)\left(\hat{V}_{A,B}(u) - \hat{V}_{A,C}(u)\right)du\right]\\ &- \mathbb{E}^Q\left[\int_t^T Z_r(t,u)\lambda_A(u)\left(\hat{V}_{A,B}(u) - \hat{V}_{B,C}(u)\right)du\right]. \end{aligned}$$

One can also rewrite Eqn.4.7 as

$$\begin{aligned}\hat{V}(t;S(t),J_A(t),J_B(t)) &= \mathbb{E}^Q\left[Z_r(t,T)\Phi(T,S(T))\right] \\ &- \mathbb{E}^Q\left[\int_t^T Z_r(t,u)\left[\lambda_A(u)+\lambda_B(u)\right]\hat{V}_{A,B}(u)du\right] \\ &- \mathbb{E}^Q\left[\int_t^T Z_r(t,u)\left[\hat{V}_{A,C}(u)\lambda_B(u)+\hat{V}_{B,C}(u)\lambda_A(u)\right]du\right].\end{aligned} \tag{4.8}$$

The last term can be interpreted as the replacement cost of replacing the derivative contract with an identical one with counterparty C. Notice, however, that there is no feasible way of knowing the new counterparty C a priori, and it will not be possible to transact at an unknown exit price $\hat{V}_{A,C}(u)$ or $\hat{V}_{B,C}(u)$. Notice also that this would have not been an issue, were we to keep the pre-crisis assumption that the inter-dealer market is made up of counterparties with (approximately) same credit quality[10], i.e. $\hat{V}_{A,C}=\hat{V}_{B,C}$.

We argue that when either

(a) there is no market mechanism that guarantees a counterparty allowing to price derivatives at zero loss - $\hat{v}_{A,B}(u)=\hat{V}(u)$, or

(b) the replacement counterparty is not known a priori and the loss amount $\hat{V}(u)-\hat{v}_{A,B}(u)$ is ill-defined,

estimation of $\hat{v}_A$ and $\hat{v}_B$ requires a **change of paradigm from replacement to recovery**, where recovery parameters for the use in pricing formulas are estimated exogenously.

Unfortunately, estimation of the recovery rates for derivative positions can itself be a convoluted one due to netting and offsetting clauses, where the actual exposures at default and their netted position are not readily (or a priori) available (see for example [Brigo and Morini 2010] and [XVA Books]). What goes pari passu with senior unsecured debt in resolution procedures is the netted cashflow on the portfolio of transactions at the time of default. Let φ_{net} be the netted expected cashflows between the counterparties, and let's denote it as v^+ when it is an asset to the counterparty A and as v^- when it is an asset to the counterparty B

$$\begin{aligned}v^+ &= max\left[\varphi_{net},0\right], \\ v^- &= min\left[\varphi_{net},0\right].\end{aligned} \tag{4.9}$$

In the recovery paradigm, where derivatives go pari passu with the senior unsecured debt (and there is no collateral), one can put

$$\begin{aligned}\hat{v}_A &= R_A v^+,\ and \\ \hat{v}_B &= R_B v^-.\end{aligned} \tag{4.10}$$

[10] [Duffie and Huang 1996] estimated that there was barely a 1bp spread that could be attributed to the bi-lateral default in interest rate swap transactions (pre-crisis).

The amount φ_{net} is the only quantity to which recovery rates R_A and R_B from the unsecured debt of counterparties can be applied.

The XVA literature largely assumes that unsecured senior debt recovery rates R_A and R_B are reasonable estimates of choice for the recovery rates χ_A and χ_B for a single derivative instrument, mainly due to the pari passu status of the latter.

On the other hand, from the standpoint of estimating recovery rates from risk-neutral prices (expectation of contractual cash flows) it is tempting to set the recovery rates for derivatives as a percentage of the pre-default value of the default-risky derivative

$$\begin{aligned} \hat{v}_A &= \chi_A \hat{V},\ and \\ \hat{v}_B &= \chi_B \hat{V}. \end{aligned} \tag{4.11}$$

This, of course, recovers the familiar expression of Eqn.3.28 for a default-risky derivative

$$\begin{aligned} \hat{V}(t; S(t), J_A(t), J_B(t)) &= \mathbb{E}^Q \left[Z_r(t,T) \Phi(T, S(T)) \right] \qquad (4.12) \\ &- \mathbb{E}^Q \left[\int_t^T Z_r(t,u) \left[(1-\chi_B) \lambda_B(u) + (1-\chi_A) \lambda_A(u) \right] \hat{V}(T-(t+u), S(u)) du \right]. \end{aligned}$$

In general, recovery rates are estimated from unsecured defaults and are counterparty specific functions of the structure and quality of defaulted counterparty's balance sheet - capital structure and residual value of assets[11], generally not observable from market prices. Collateral agreements can be put in place to make the recovery levels as predictable as possible. Collateralization does not have to make the recovery rate equal to one. Estimation of recovery rates is exogenous to pricing processes.

For the **recovery paradigm with collateralization** recovery rate χ for a derivative has to be adjusted for collateral to make the estimated recovery rate that of an unsecured senior debt, so that at default the mark-to-market values $\hat{V}$ can go pari passu with the unsecured senior debt.

In the recovery paradigm with collateralization one can estimate $\hat{v}_A$ and $\hat{v}_B$ as follows. If we assume netted collateral posting and collection, then there is a $C(u)$ amount of collateral available to the solvent party at the time of default. If the position is over-collateralized - $\hat{V}(u) < C(u)$, then recovery is the value of the derivative, otherwise it is the collateral amount plus recovery from the unsecured portion - $C(u) + \chi_{A,B}(\hat{V}(u) - C(u))$. This can be presented in a combined manner as

[11] Strictly speaking, because of the exogenous nature of this process it is difficult to refer to Eqn.4.12 as risk-neutral price, unless its second term (or its components) is implied from market prices (see also [Brigo 2018] and [Risk.net 2017]), analogous to bond markets, where the expected loss premium can be implied by the difference between default-risky and default-risk-free bond prices.

follows[12]

$$\begin{aligned}\hat{v}_A &= (\hat{V}-C)^- + \chi_A(\hat{V}-C)^+ + C,\\ \hat{v}_B &= (\hat{V}-C)^- + \chi_B(\hat{V}-C)^+ + C. \end{aligned} \tag{4.13}$$

Here $x^+ = max(x,0)$ and $x^- = min(x,0)$.

This leads to **collateral-adjusted derivatives recovery rates** $\tilde{\chi}_A$ and $\tilde{\chi}_B$

$$\begin{aligned} 1-\tilde{\chi}_A(C_A) &= -\frac{\Delta\hat{V}_A}{\hat{V}} = \frac{\hat{V}-(\hat{V}-C)^- - \chi_A(\hat{V}-C)^+ - C}{\hat{V}} = (1-\chi_A)\frac{\left(\hat{V}-C\right)^+}{\hat{V}}\\ &= (1-k)^+(1-\chi_A), \end{aligned} \tag{4.14}$$

$$\begin{aligned} 1-\tilde{\chi}_B(C_B) &= -\frac{\Delta\hat{V}_B}{\hat{V}} = \frac{\hat{V}-(\hat{V}-C)^+ - \chi_B(\hat{V}-C)^+ - C}{\hat{V}} = (1-\chi_B)\frac{\left(\hat{V}-C\right)^+}{\hat{V}}\\ &= (1-k)^+(1-\chi_B), \end{aligned} \tag{4.15}$$

with $k = \frac{C}{\hat{V}}$ as the level of collateralization for the transaction. This also makes the hedge ratios in Eqn.3.18 and Eqn.3.19 explicitly proportional to the non-collateralized portion of the derivative

$$\tilde{h}^c_A = -(1-k)^+ z_A \frac{h'_{\hat{V}}\hat{V}}{P_A}, \tag{4.16}$$

$$\tilde{h}^c_B = -(1-k)^+ z_B \frac{h'_{\hat{V}}\hat{V}}{P_B}. \tag{4.17}$$

Thus, one arrives to **the collateral-adjusted form** of the main equation Eqn.3.26 for the price of a collateralized default-risky derivative

$$\begin{aligned}\mathcal{L}^{(r-\delta)}\hat{V} - r\hat{V} &= \left[(1-\tilde{\chi}_A)\lambda_A + (1-\tilde{\chi}_B)\lambda_B\right]\hat{V}\\ &= \left[(1-\chi_A)\,\lambda_A + (1-\chi_B)\,\lambda_B\right](1-k)^+\,\hat{V}, \end{aligned} \tag{4.18}$$

with the corresponding Feynman-Kač solution

$$\begin{aligned}\hat{V}(t;S(t),J_A(t),J_B(t)) &= \mathbb{E}^Q\left[Z_r(t,T)\Phi(T,S(T))\right] \qquad (4.19)\\ &- \mathbb{E}^Q\left[\int_t^T Z_r(t,u)\left[(1-\chi_B)\,\lambda_B + (1-\chi_A)\,\lambda_A\right](1-k)^+\,\hat{V}(u)du\right],\ \textit{or}\end{aligned}$$

[12]Note that these are not the recovery expressions widely used in the XVA literature. Here $\hat{v}_{A,B}(\chi_{A,B}, C=0) = \chi_{A,B}\hat{V}$ as in Eqn.4.11, because $\hat{V}$ as a price is kept positive. It does not carry an asset ("+") or liability ("-") sign implicitly.

$$\begin{aligned}\hat{V}(t; S(t), J_A(t), J_B(t)) &= \mathbb{E}^Q\left[Z_r(t,T)\Phi(T,S(T))\right] \\ &- \mathbb{E}^Q\left[\int_t^T Z_r(t,u)\left[(1-\chi_B)\lambda_B + (1-\chi_A)\lambda_A\right]\left(\hat{V}(u) - C(u)\right)^+ du\right]\end{aligned} \tag{4.20}$$

If the posted collateral is not allowed to be netted (e.g., when collateralization includes initial margins) the collateralization level k acquires a counterparty subscript - $k_A = \dfrac{C + I_A}{\hat{V}}$ and $k_B = \dfrac{C + I_B}{\hat{V}}$, where the level of collateralization includes the initial margin collateral amounts I_A and I_B.

With a non-netted initial margin the Feynman-Kač expression Eqn.4.20 obtains the following form[13]

$$\begin{aligned}\hat{V}(t; S(t), J_A(t), J_B(t)) &= \mathbb{E}^Q\left[Z_r(t,T)\Phi(T,S(T))\right] \\ &- \mathbb{E}^Q\left[\int_t^T Z_r(t,u)\,(1-\chi_B)\,\lambda_B\left(\hat{V}(u) - C(u) - I_B(u)\right)^+ du\right] \\ &- \mathbb{E}^Q\left[\int_t^T Z_r(t,u)\,(1-\chi_A)\,\lambda_A\left(\hat{V}(u) - C(u) - I_A(u)\right)^+ du\right].\end{aligned} \tag{4.21}$$

In summary, the expression Eqn.4.21 for a no-arbitrage price of a default-risky collateralized derivative implies that:

- **no-arbitrage prices for default-risky derivatives should price the unsecured portion of the default-risky derivative, with unsecured recovery rates**, if they are to be pari passu with the senior unsecured debt,
- **applying senior unsecured debt recovery rates to pricing of a single derivative is generally not accurate,** as these rates should be applied to the closeout netted exposures at default that go pari passu to senior unsecured debt (an exception could be the simple case where exposures are unilateral),
- **the collateral cannot be part of dynamic variables in the replicating portfolio**, either as part of the risky positions (the hedge) because it prices no risk factor, or the money account because it's accruing at a non-risk-free rate,
- **collateral is part of an exogenously estimated parameter - the recovery rate**, which is generally dependent on the capital structure and the residual value of the assets, both unobservable to the market.

The last two points on collateral make intuitive sense.

[13] More accurately, though, cash collateral amounts $C(u)$ and $I_{A,B}(u)$ at the intermediate times u are from the previous time interval $u - \delta u$ (See also [Pykhtin ét al]).

There were no market-priced securities in the replicating portfolio that priced the recovery risk directly in all states of the world. Stated otherwise, the self-financing replicating portfolio was set up in a market that did not have securities that were perfectly correlated with a recovery risk factor. For the same reason it does not help introducing new dynamic risk factors to model the recovery rates χ_A and χ_B. This constitutes an incomplete market, causing semi-replication. Such risks can only be mitigated (collateralization and/or guarantees) through means outside the market transactions[14], not fully hedged.

This semi-replication should not be confused with the case in [Burgard and Kjaer 2013]. The latter approach cannot be called a semi-replication as it simply is a voluntary under-hedging of the default risk of the issuer of the derivative by choosing different from the full replication weights. In other words, the market in [Burgard and Kjaer 2013] is still complete[15].

This is easily observed if one rewrites the full replication weights $\tilde{h}$ in Eqn.3.17 - 3.19 in notations of [Burgard and Kjaer 2013]

$$\begin{aligned}
h_{A,B}P_{A,B} - R_{A,B}h_{A,B}P_{A,B} \quad &:= \quad P_{A,B} - P^D_{A,B},\ with\ h^{'}_{\hat{V}} = 1,\ leading\ to \\
\tilde{h}_S = -\Delta h^{'}_{\hat{V}},\ and\ \tilde{h}_{\hat{V}} = h^{'}_{\hat{V}}; \quad &\Longleftrightarrow \quad \tilde{h}_S = -\Delta,\ and\ \tilde{h}_{\hat{V}} = 1; \qquad (4.22) \\
\tilde{h}_A = -z_A\frac{\hat{V}h^{'}_{\hat{V}}}{P^-_A} = \frac{(1-\chi_A)\hat{V}}{(1-R_A)P^-_A}h^{'}_{\hat{V}}, \quad &\Longleftrightarrow \quad P_A - P^D_A = -\Delta\hat{V}_A = \hat{V} - g_A; \qquad (4.23) \\
\tilde{h}_B = -z_B\frac{\hat{V}h^{'}_{\hat{V}}}{P^-_B} = \frac{(1-\chi_B)\hat{V}}{(1-R_B)P^-_B}h^{'}_{\hat{V}}, \quad &\Longleftrightarrow \quad P_B - P^D_B = -\Delta\hat{V}_B = \hat{V} - g_B. \qquad (4.24)
\end{aligned}$$

[Burgard and Kjaer 2013] choose $V - g_A = FullHedgeWeight - \epsilon$, "taking inspiration from funding considerations". Not choosing the full replication weights $\tilde{h}$ creates an arbitrage opportunity between holding a derivative position of a counterparty against a portfolio of counterparty's bonds.

There is also no reason that would follow from no-arbitrage pricing for the specific choices of the bonds and their recovery rates in [Burgard and Kjaer 2013] (zero recovery for one counterparty and subordinated debt for the other). The full replication is achievable with any portfolio of counterparty A bonds. If we are interpreting the recovery rates R_A and R_B as the recovery percentage per dollar of a senior unsecured exposure, and the derivative positions go pari passu with the unsecured senior debt of the defaulted counterparty (which they do), then one puts $\chi_A = R_A$ and $\chi_B = R_B$ and arrives to hedge ratios with $z_A = z_B = 1$. If the counterparty defaults are hedged using any other combination of the defaulted counterparty's bonds, then the hedge ratios are adjusted accordingly. For example, for hedging with subordinated debt one has $z_{A,B} < 1$ for the hedge ratios since

[14] Or, perhaps, capitalization with a bank's balance sheet, to complete the market.

[15] A better terminology probably would be incomplete replication in complete markets, as opposed to semi-replication in incomplete markets.

subordinated debt recovers less than the senior debt to which derivatives are pari passu. We discuss the approach in [Burgard and Kjaer 2013] in more detail in Appendix B, as it is used widely for deriving XVA expressions (see, for example [Green and Kenyon 2015]).

Finally, it is worth noting, that **collateralization can also play a systemic role**. One could **redefine closeout rules to limit the entitlement of the derivatives recovery exclusively to the recovery from collateral accounts**, with no further recourse to the assets of the defaulted counterparty. This would decouple derivatives trading from the rest of bank's deposit funded balance sheet (e.g., no ring-fencing would be necessary).

5 Closing Remarks

Following the approach in [Piterbarg 2010] and [Burgard and Kjaer 2013] market participants have generated adjustments to derivatives pricing formulas to reflect funding and hedging costs, collectively referred to as XVAs. In a separate effort [Brigo ét al 2017] show that these costs can also be recovered if they are assigned as dividends to the replicating market securities. However, in either approach there seems to be no market that would clear these dividend or cost components as part of no-arbitrage prices. In either case requiring XVAs to be included into a price of a derivative effectively amounts to what's referred to as "donations" in [Andersen, Duffie and Song 2017].

In this paper we have shown that XVAs do not originate from no-arbitrage pricing[16] as they are not part of a self-financing replicating portfolio of market traded securities (there is no market that clears the prices with XVAs as dividends).

Recently, there have been notable attempts to bring XVAs into the corporate finance and accounting frameworks ([Andersen, Duffie and Song 2017], [Albanese ét al 2014, 2015, 2017] and [Kjaer 2017]).

In a forthcoming paper [Tumasyan 2019] we will discuss these efforts and will formulate an approach for recovering **XVAs as P&L measures of balance sheet consumption** for a derivative transaction on a bank's balance sheet, with no reference to no-arbitrage (risk-neutral) pricing.

We will also argue, that **adding XVAs as costs to the price of a derivative transforms derivatives from a market traded instrument into a banking instrument** (contractual cash flows discounted by an all-in yield).

[16] Although formally, setting the second and third terms in Eqn.4.21 equal to $CVA + DVA$ recovers some members of the familiar XVA family.

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Appendix

A Money Accounts in [Piterbarg 2010] & [Burgard and Kjaer 2010]

We explicitly expand the money accounts in [Piterbarg 2010] and [Burgard and Kjaer 2010].

To arrive to the results of [Piterbarg 2010]

$$\mathcal{L}V - \delta S\frac{\partial V}{\partial S} = -r_R S\frac{\partial V}{\partial S} + r_C C + r_F(V - C) \tag{A.1}$$

one has to solve the z.i.i. constraint (Eqn.2.8) with $h_{\hat{V}} = -1$, and add and subtract the collateral account

$$M = -\Delta S + (V - C) + C. \tag{A.2}$$

The addition and subtraction of the collateral account generates the money account components M_R, M_F and M_C which are then assigned accrual rates $r_a, a = C, F, R$, motivated by the following cost structure:

$M_R = -\Delta S$ amount of the underlying security borrowed at the repo rate r_R, with dividend income of δ;

$$dM_R = (r_R - \delta)M_R dt;$$

$M_F = V - C$ amount to be borrowed/lent unsecured from the treasury desk for collateral, which accrues at the funding rate of r_F

$$dM_F = r_F M_F dt;$$

$M_C = C$ the collateral account that accrues at the collateral rate of r_C;

$$dM_F = r_C M_C dt.$$

For the money account in [Piterbarg 2010] the standard no-arbitrage conditions in Eqn.2.5 lead to

$$\mu_V - (\mu + \delta)\frac{\sigma_V}{\sigma} = \frac{1}{V}\sum_{a=C,F,R} M_a r_a,$$

which means that one needs to put

$$\begin{aligned}\frac{1}{V}\sum_a M_a r_a + \frac{\sigma_V}{\sigma} r &\stackrel{?}{=} r, \; or \\ \frac{1}{V}\sum_a M_a r_a &\stackrel{?}{=} r\left(1 - \frac{\sigma_V}{\sigma}\right) = r\frac{V - \Delta S}{V}, \; or \\ \sum_{a=C,F,R} \frac{M_a}{V - \Delta S} r_a &\stackrel{?}{=} r. \end{aligned} \tag{A.3}$$

This means (the z.i.i. constraint Eqn.A.2 holds)

$$\sum_{a=C,F,R} \frac{M_a}{\sum_{a=C,F,R} M_a} = \sum_{a=C,F,R} w_a = 1, \tag{A.4}$$

and consequently for Eqn.A.3

$$\sum_{a=C,F,R} w_a r_a \stackrel{?}{=} r, \; or \sum_{a=C,F,R} w_a \left(r_a - r\right) \stackrel{?}{=} 0. \tag{A.5}$$

Eqn.A.5 states that the "portfolio of funding accounts" that gives the money account a structure in [Piterbarg 2010] has to be a risk-free (or zero beta) portfolio.

The arguments above apply to the case of [Burgard and Kjaer 2010] with the following definitions for the money account components.

- Split the funding component M_F of the money account into two pieces to account for any surplus or shortfall cash held by the seller after the own bonds have been purchased:

  $\left(-\hat{V} - \Delta\hat{V}_B\right)^+$ surplus cash held by the seller after the own bonds have been purchased accruing at risk-free rate r;

  $$dM_F^+ = rM_F^+ dt;$$

  $\left(-\hat{V} - \Delta\hat{V}_B\right)^-$ shortfall that needs to be funded through borrowing, at the financing rate of r_F

  $$dM_F^- = r_F M_F^- dt;$$

- $M_R = -\Delta S$ an account for the underlying security borrowed, accruing at the repo rate q, provides a dividend income of δ;

  $$dM_R = (r_R - \delta)M_R dt;$$

- M_C an account for the proceeds of shorting the counterparty bond through a repurchase agreement at rate r (it is assumed that the haircut in this repo is zero, so that the repo rate for the counterparty bond can be replaced with a risk-free rate)

$$dM_C = rM_C dt.$$

With these notations equations Eqn.A.3 - Eqn.A.5 follow for the case of [Burgard and Kjaer 2010].

B The Case of Semi-Replication in [Burgard and Kjaer 2013]

We discuss the semi-replication approach introduced in [Burgard and Kjaer 2013], since this approach is used widely for deriving KVA and MVA expressions (as in [Green and Kenyon 2015]). The approach cannot be called a semi-replication as it simply amounts to under-hedging the default risk of the "issuer" of the derivative by choice.

We will apply the no-arbitrage pricing approach in the main body of the paper to [Burgard and Kjaer 2013], to show that the semi-replication in this paper allows arbitrage.

B.1 Semi-Replication - a Misnomer

[Burgard and Kjaer 2013] set up a "hedge portfolio" Π (not the same as Π^h above) as

$$\Pi(t) = q(t) \cdot h(t) = h_S S + h_A P_A + h_B P_B + M - C \tag{B.1}$$

with a strategy that $V + \Pi = 0$.[17]. Then $\Psi := V + \Pi$ is the arbitrage portfolio Π^h above with $h_V = 1$.

[Burgard and Kjaer 2013] introduce a money account distribution as $M = M_S + M_B$, where M_S and M_B are assumed to be "financing" the S and P_B positions respectively. The latter interpretation provides the justification for extra constraints

$$\begin{aligned} h_B P_B + M_B &= 0 \\ h_S S + M_S &= 0. \end{aligned} \tag{B.2}$$

With Eqn.B.2 the z.i.i. constraint for Ψ looks as follows

$$\Psi = V + \Pi = V + \underbrace{h_S S + M_S}_{=0} + h_A P_A + \underbrace{h_B P_B + M_B}_{=0} - C = V + h_A P_A - C = 0. \tag{B.3}$$

[17] "...except, possibly, at issuer default" -note by the authors.

Using Eqn.B.2, the dynamics of the money accounts M_S and M_B are assigned rates of return along lines of the financing arguments, with q and q_B the repo rates for financing the S and P_B positions:

$$\begin{aligned} dM_S &= rM_S dt = -rh_S S dt \longrightarrow -(q-\delta)\, h_S S dt; \\ dM_B &= rM_B dt = -rh_B P_B dt \longrightarrow -q_B h_B P_B dt. \end{aligned} \tag{B.4}$$

[Burgard and Kjaer 2013] then write the instantaneous return for the self-financing portfolio Ψ as

$$d\Psi = dV + d\Pi = dV + h_S dS + h_A dP_A + h_B dP_B + dM_S + dM_B - dC. \tag{B.5}$$

Plugging in the price dynamics into Eqn.B.5 and using the notations from [Burgard and Kjaer 2013] for $h_{A,B}P_A - R_A h_{A,B} P_A := P_{A,B} - P^D_{A,B}$ leads to

$$\begin{aligned} d\Psi &= dV + h_S dS + \left[r_A P_A + h_B\,(r_B - q_B)\,P_B + (\delta - q)\,h_S S - r_C C\right] dt \\ &+ \left[P^D_A - P_A\right] dJ_A + \left[P^D_B - P_B\right] dJ_B. \end{aligned} \tag{B.6}$$

Now, recalling Eqn.B.3 and choosing $R_B = 0 \Rightarrow P^D_B = 0$ leads to the following expression for $d\Psi$[18]

$$\begin{aligned} d\Psi = dV + d\Pi &= \left[\mu_V V + h_S \mu S + r_A P_A + h_B\,(r_B - q_B)\,P_B + (\delta - q)\,h_S S - r_C C\right] dt \\ &+ \left[\sigma_V V + h_S \sigma S\right] dz \\ &+ \left[P^D_A - C + g_A\right] dJ_A + \left[\Delta V_B - h_B P_B\right] dJ_B. \end{aligned} \tag{B.7}$$

where we have used $\Delta V_A = g_A - V$.

One can of course require that all the terms in Eqn.B.7 be set equal to zero to avoid arbitrage, however, we will follow the paper by setting

$$h_S = -\frac{\sigma_V}{\sigma}\frac{V}{S} = -\Delta, \tag{B.8}$$

$$h_B = \frac{\Delta V_B}{P_B}, \; and \tag{B.9}$$

$$h_A = P^D_A - C + g_A = \epsilon \neq 0. \tag{B.10}$$

With these new notations, for the delta-hedged portfolio $\Psi = V + \Pi$ one can write

$$\begin{aligned} d\Psi = dV + d\Pi &= \left[\mu_V V - \Delta\mu S + r_A P_A + (r_B - q_B)\,\Delta V_B - (\delta - q)\,\Delta S - r_C C\right] dt \\ &+ \left[P^D_A - C + g_A\right] dJ_A. \end{aligned} \tag{B.11}$$

Using the expression for $\mu_V V$, adopting the notations $r_B - q_B = \lambda_B$[19], $s_C = r_C - r$, $\epsilon = P^D_A - C + g_A$ and using the remaining z.i.i constraint Eqn.B.3 one arrives to the following main expression of [Burgard and Kjaer 2013]

$$\begin{aligned} d\Psi = dV + d\Pi &= \left[\mathcal{L}^{q-\delta} V - s_C C - (r + \lambda_A + \lambda_B)\,V + (g_A - \epsilon)\lambda_A + g_B \lambda_B\right] dt \\ &+ \epsilon dJ_A. \end{aligned} \tag{B.12}$$

[18] Notice, that at this point there is no particular reason for choosing $R_B = 0$.

[19] Note that λ here is different from the one in previous sections, it is defined with respect to q_B.

[Burgard and Kjaer 2013] state that "We assume that the issuer wants the strategy described above to evolve in a self-financed fashion while he is alive", and that "This implies that the issuer requires the total drift term of $dV + d\Pi$ to be zero."

This produces a PDE in [Burgard and Kjaer 2013] from which XVA expressions in the literature are derived and interpreted by others (e.g., [Green and Kenyon 2015]).

Let's notice now that the no-arbitrage conditions for the self-financing portfolio $\Psi = V + \Pi$ with $\Psi(t) = 0$ would imply that arbitrage exists even if one sets the drift term in Eqn.B.12 to zero, since the probability of default for the counterparty A is non-zero - $Prob[dJ_A = 1] > 0$, and there is always an ϵ such that

$$\mathbb{P}\left[\Psi(t+dt) > 0\right] = \mathbb{P}\left[\epsilon dJ_A > 0\right] \quad > \quad 0, \; and \tag{B.13}$$

$$\mathbb{P}\left[\Psi(t+dt) \geq 0\right] = \mathbb{P}\left[\epsilon dJ_A \geq 0\right] \quad = \quad 1. \tag{B.14}$$

Moreover, due to the "abridged" version of the z.i.i. constraint Eqn.B.3

$$\begin{cases} \Psi = \hat{V} + P_A^- - C = 0 \\ \quad g_A + P_D - C = \epsilon. \end{cases} \Longrightarrow \hat{V} - g_A + P_A^- - P_D = -\epsilon.$$

one can write

$$\left[g_A - \hat{V} + P_A^D - P_A^-\right] dJ_A = \epsilon dJ_A \neq 0. \tag{B.15}$$

Eqn.B.15 means that there is an arbitrage opportunity in holding the derivative position vs. the positions in bonds, or vise-versa.

The reason for this is, of course, clear - it is due to the choice of hedge ratios Eqn.B.8 , instead of the full hedge ratios $\tilde{h}_{A,B}$ in Eqn.3.18 and Eqn.3.19. We can rewrite them as

$$\tilde{h}_B\left(P_B^- - P_B^D\right) - \Delta\hat{V}_B h_{\hat{V}} = \tilde{h}_B\left(P_B^- - P_B^D\right) + \left(\hat{V}_B - g_B\right) h_{\hat{V}} \quad = \quad 0, \; and \tag{B.16}$$

$$\tilde{h}_A\left(P_A^- - P_A^D\right) - \Delta\hat{V}_A h_{\hat{V}} = \tilde{h}_A\left(P_A^- - P_A^D\right) + \left(\hat{V}_A - g_A\right) h_{\hat{V}} \quad = \quad 0. \tag{B.17}$$

to show, that the semi-replication in [Burgard and Kjaer 2013] simply generates a shift of ϵ with respect to the full replication hedge ratio for the counterparty A

$$\hat{V} - g_A = -\left(P_A^- - P_D\right) - \epsilon = Full\ Replication\ Weight - \epsilon. \tag{B.18}$$

To summarize:

- the ϵ shift off of the full replication ratio for the counterparty A does not come from any no-arbitrage constraints,
- it is simply (voluntary) under-hedging and not a semi-replication.

Furthermore, there is no reason for the specific choice of the bonds in [Burgard and Kjaer 2013].

B.2 Semi-replication - Choice of Replicating Bonds

There is no reason for the specific choices of the bonds and their recoveries in [Burgard and Kjaer 2013], the full replication is achievable with any portfolio of counterparty A bonds.

To see this we will approach it from a more general setting. Assume the counterparty A above has issued n bonds $P^-_{A,i}$ with different seniorities - different recovery rates $R_{A,i}$. One can write for these bonds

$$dG_{A,i} = dP^-_{A,i} + r_{A,i}P^-_{A,i}dt = -(1 - R_{A,i})P^-_{A,i}dJ_A + r_{A,i}P^-_{A,i}dt. \tag{B.19}$$

This transforms the last term of Eqn.3.7 into

$$\left[h_{\hat{V}}\Delta\hat{V}_A - \sum_{i=1}^{n} h_{A,i}(1 - R_{A,i})P^-_{A,i}\right] dJ_A = \left[h_{\hat{V}}\Delta\hat{V}_A + \left(P^D_A - P^-_A\right)\right] dJ_A. \tag{B.20}$$

Here the notations are generalization of the ones used in [Burgard and Kjaer 2013]

$$-\sum_{i=1}^{n} h_{A,i}P^-_{A,i} + \sum_{i=1}^{n} h_{A,i}R_{A,i}P^-_{A,i} = P^D_A - P^-_A. \tag{B.21}$$

Setting also $h_{\hat{V}} = 1$ and $\Delta\hat{V}_A = g_A - \hat{V}$ as in [Burgard and Kjaer 2013] one can write

$$\left[\Delta\hat{V}_A + \left(P^D_A - P^-_A\right)\right] dJ_A = \left[g_A - \hat{V} + P^D_A - P^-_A\right] dJ_A. \tag{B.22}$$

It is easy to see that weights $h_{A,i}$ exist such that the expression in square brackets of Eqn.B.22 can be set to zero

$$g_A - \hat{V} + P^D_A - P^-_A = 0, \ \textit{or} \ \hat{V} - g_A = -\left(P^D_A - P^-_A\right), \tag{B.23}$$

i.e. the loss in value of the derivative due to the default by counterparty A is restored by the recovery from a short position in the portfolio of bonds issued by the counterparty A.

Notice, that it is not the individual weights in the portfolio of bonds (consequently, not the seniority or other characteristics of each of the issue) that matter, but rather the total amount of the debt holdings

$$\begin{aligned} -\sum_{i=1}^{n} h_{A,i}P^-_{A,i} + \sum_{i=1}^{n} h_{A,i}R_{A,i}P^-_{A,i} &= -h_A\sum_{i=1}^{n} v_{A,i}P^-_{A,i} + h_A\sum_{i=1}^{n} v_{A,i}R_{A,i}P^-_{A,i} \\ &= h_A\left(P^D_A - P^-_A\right). \end{aligned} \tag{B.24}$$

Here $h_A = \sum_{i=1}^{n} h_{A,i}$ and $v_{A,i} = \dfrac{h_{A,i}}{h_A}$.